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TITLE- Apollo CM Water Landing Acceleration Probabilities

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AUTHOR(S)- G. R. Andersen

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ABSTRACT

An Apollo CM is considered to be in vertical descent. It is suspended by three parachutes in such a way that its main axis, when in equilibrium, makes an angle of θ_0 with the vertical. However, as the CM falls it oscillates in a random manner about θ_0 and twists randomly relative to the vertical. This behavior is combined with a statistical description of the ocean and an experimentally determined function giving the acceleration along the main axis of the CM upon impact with the ocean to yield an estimate of the probability that this acceleration will exceed a given number of g's. It is found that the normal mission limit is exceeded less than 5% of the time; while the probability of exceeding the emergency mission limit is essentially zero. A more detailed summary of results is given in Section 6.0 and Appendix II.

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BELLCOMM, INC.

SUBJECT: Apollo CM Water Landing Acceleration Probabilities - Case 330

DATE: March 15, 1967

FROM: G. R. Andersen

TM-67-1033-1

TECHNICAL MEMORANDUM

INTRODUCTION

The purpose of this paper is to determine the probability with which the acceleration along the main axis of the Apollo CM will exceed a given value upon impact with the ocean. This information was requested in connection with the Apollo CM analysis of the couch stroke required for a water landing. [2]

In Sections 1.0 through 5.0 a mathematical idealization of the problem is constructed based on accepted statistical descriptions of the wind speed, wave slope and water particle velocity for the mid-Pacific in the month of February [1] together with an experimentally determined approximation to the acceleration function.

The operational conclusions obtained from this study are given in Section 6.0 and Appendix II in the form of the conditional probability distribution of the acceleration given that the angle of entry of the capsule into the water is between zero and 27° and a certain constraint is imposed on the water particle velocity. Also it is shown in Section 6.0 that this conditional probability distribution can be considered to be an upper bound on the unconditional distribution of the acceleration.

1.0 PHYSICAL FORMULATION

Test drops of the CM suspended by three parachutes have shown that the CM axis, C_A, is randomly oriented with respect to the wind and that it oscillates sinusoidally about a nominal suspension angle, θ_0, with an oscillation amplitude of 4°. We will assume that this oscillation takes place in a plane containing the CM axis, C_A, and the extended vertical through the point of suspension of the chutes and the capsule.

Let γ denote the angle of entry of the capsule into the water. That is, γ is the angle between the vector, N_W, normal to the wave front (struck by the capsule) and the capsule axis, C_A.

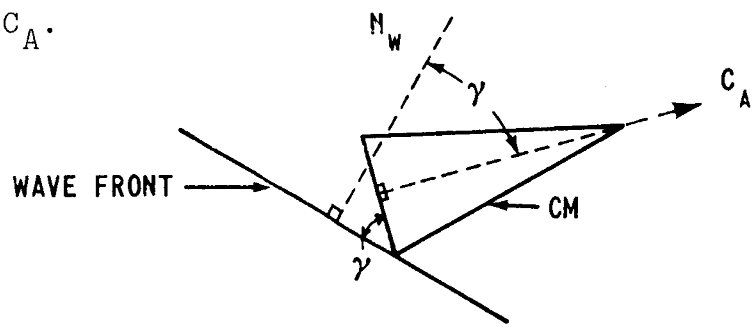

FIGURE 1
(IN TWO DIMENSIONS)

If we let ϕ denote the orientation angle of the CM (with respect to the horizontal projection of C_A), θ the wave slope and ψ the nominal suspension θ_0 plus the sinusoid motion, α, of the capsule, then in place of Figure 1 we have the following three-dimensional representation of our model:

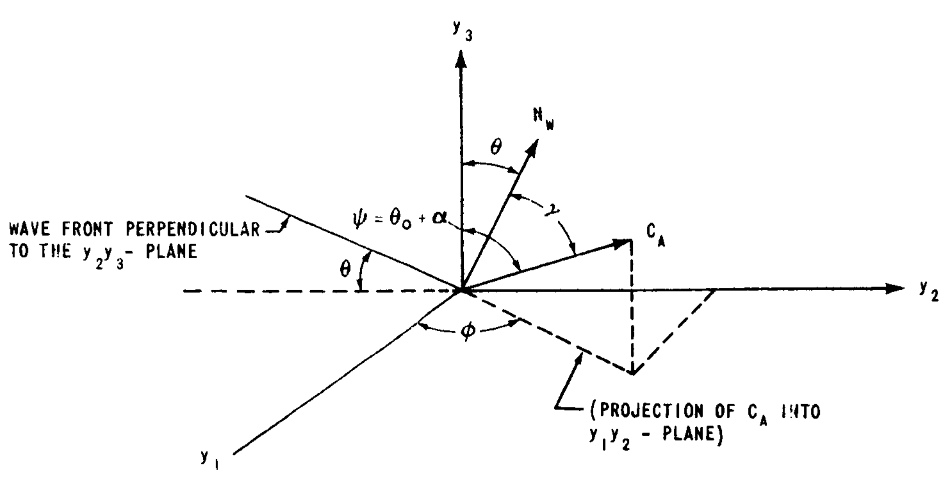

FIGURE 2

In order to describe the acceleration along the axis C_A, three additional factors must be considered. One of these is the wind speed, v, at the moment of impact. This is obvious since the wave slope is clearly a function of the wind speed, $\theta = \theta(v)$. Another is the so-called water particle velocity, W. Since W is also a function of the wind speed we will write $W = W(v)$.

In the model (cf. Figure 2) the wave slope $\theta = \theta(v)$ will always be measured in positive units and so will not indicate whether the capsule strikes the wave on its up or downwind side. This distinction is introduced into the model by allowing the water particle velocity to take both positive and negative values; that is, when $W < 0$ the wave "front" in Figure 2 will denote the downwind side of the wave and when $W > 0$ it will denote the upwind side.

The third additional factor is the vertical velocity V at impact of the descending CM. It is assumed that V is constant (independent of the wind speed). In the numerical calculation we will set $v = 28$ fps.

The acceleration function along the CM axis, C_A, upon impact is denoted by $\ddot{x}$. Based on the results of 1/4 scale model CM test drops conducted by NASA, Langley, [2] we may assume that

$$\ddot{x} = \ddot{x}(w, \gamma) = \ddot{x}(w(v), \gamma(v))$$

is a function of the water particle velocity w, the entry angle γ and the wind speed v through w and γ. That γ is function of the wind speed is clear from Figure 2. Indeed,

$$\gamma = \gamma(\theta, \phi, \alpha) = \gamma(\theta(v), \phi, \alpha) \quad ,$$

where v again denotes the (observed) wind speed.

Thus,

$$\ddot{x} = \ddot{x}(w(v), \gamma(\theta(v), \phi, \alpha)) \quad . \tag{1}$$

Now, as one would surmise, we are <u>not</u> given an explicit functional form for $\ddot{x}$, but rather a graph (Appendix I) of this function for particular values of $w = w(v)$. In order to obtain a solution of the problem, the curve $\ddot{x}(0,\gamma)$ was approximated from above by a function of the cosine of the entry angle. We consider this approximation as the function itself and write

$$\ddot{x}(0,\gamma) = 38(\lambda(\gamma))^{14} \quad , \tag{2}$$

where

$$\lambda = \lambda(\gamma) = \cos \gamma$$

for $0 \leqq \gamma \leqq 27°$ (the extent of the data).

This combined with the <u>empirical</u> formula (furnished by J. Richey, Bellcomm):

$$\ddot{x} = \ddot{x}(w,\gamma) = \left(1 + \frac{w}{V}\right)^{1.3} \ddot{x}(0,\gamma) \quad ,$$

where V is the vertical velocity at impact of the descending CM (V = 28 fps as mentioned earlier), leads to the tractable form:(1)

$$\begin{aligned} \ddot{x} &= \ddot{x}(w(v), \lambda(v)) \\ &= 38 \left(1 + \frac{w(v)}{V}\right)^{1.3} (\lambda(v))^{14} \end{aligned} \qquad (3)$$

for $|w(v)| \leqq 3\sigma^*$ (σ^* will be defined below in such a way that $\sigma^* < 5$ for all probable values of v) and $\cos 27^\circ \leqq \lambda(v) \leqq 1$, where

$$\lambda = \lambda(v) = \cos \gamma(v) \quad . \qquad (4)$$

From Figure 1 we easily see that

$$N_W = (0, \sin \theta, \cos \theta)$$

and

$$C_A = (\sin \psi \cos \phi, \sin \psi \sin \phi, \cos \psi) \quad .$$

Hence, since $|N_W| = 1 = |C_A|$, we have

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(1)From this point on the dependence of the acceleration $\ddot{x}$ on the entry angle will be given through its dependence on $\lambda = \lambda(\gamma) = \cos \gamma$.

$$\left.\begin{aligned}\lambda &= \cos\ \gamma = N_W \cdot C_A \\ &= \sin\ \theta\ \sin(\theta_0+\alpha)\ \sin\ \phi + \cos\ \theta\ \cos(\theta_0+\alpha)\end{aligned}\right\} \quad (5)$$

where we have used $\psi = \theta_0 + \alpha$.

2.0 PROBABILISTIC FORMULATION

Physically, the function $\ddot{x} = \ddot{x}(w,\lambda)$ is defined in the entire set $R_1 x[-1,1]$, although it is only given (equation 3) for (w,λ) in subsets of the form

$$C(\delta_1) = \left\{(w,\lambda)\varepsilon R_2 : \delta_1 \leq \lambda \leq 1 ; |w| \leqq 3\sigma^*\right\} \quad (6)$$

for all δ_1 such that $\cos 27° \leqq \delta_1 < 1$.

However, the geometry of the model indicates that the larger values of $\ddot{x}$ will occur when the entry angle γ is small; that is, when the cosine of the entry angle, λ, is close to 1. Also the restraint on the water particle velocity, w, (viz., $|w| < 3\sigma^*$) will merely reflect the fact that the water particle velocity is constrained (in probability), to a symmetric interval about its mean value zero of at least three standard deviations in length on each side of the origin. Hence, for the purposes of the problem and in view of the limited amount of data available we will find the probability that acceleration exceeds a given value ξ given that the entry angle and water particle velocity take values in the set $C(\delta_1)$,($\cos 27° \leqq \delta_1 < 1$). That is, we will find

$$P\left\{[\ddot{x}(\underline{W},\underline{\lambda})>\xi]\,\middle|\,[(\underline{W},\underline{\lambda})\varepsilon C(\delta_1)]\right\}^{(2)} \tag{7}$$

3.0 ASSUMPTIONS

The verbal assumptions of the introduction may be stated mathematically as follows:

(i) $\underline{\alpha} = \alpha_0 \sin \underline{\beta}$, where $\alpha_0 = 4°$ and $\underline{\beta}$ is a random variable (r.v.) having a uniform distribution over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) $\underline{\phi}$, the orientation angle, is a r.v. having a uniform distribution over the interval $(0, 2\pi)$.

(iii) $\underline{U}$, the wind speed, is a r.v. with distribution function (d.f.) G, where G is given in an MSC report [1] (also see Appendix IV).

(iv) θ, the wave slope, is a r.v. whose (conditional) d.f. given that the r.v. $\underline{U}$ assumes the value v (the observed wind speed in knots) is the same as the d.f. of the absolute value of the Gaussian r.v. $\theta^*(v)\varepsilon N(0,\sigma_S(v))$; that is, $\theta^*(v)$ has a Gaussian distribution with zero mean and standard deviation $\sigma_S(v)$, given by $\sigma_S(v) = 0.79[0.808(10^{-3})v-0.0058]^{1/2}$ radians for $v > v_0 = (5.8/.808)$knots . [1]

---

(2)We will distinguish between a random variable and the values assumed by a random variable by writing $\underline{\lambda}$ for the former and λ for the latter. When convenient we will also use the corresponding capital letter for the r.v. and lower case for the number; e.g., $\underline{W}$ and w.

(v) W, the water particle velocity, is a random variable whose (conditional) d.f. given that the wind speed $\underline{U}$ assumes the value v is identical with an $N(0, \sigma_W(v))$ d.f., where

$$\sigma_W^2(v) = \frac{\alpha v^2}{2}\sqrt{\frac{\pi}{\beta}}\left\{1 - \Phi\left(\frac{13}{\pi}\sqrt{2\beta}\,\frac{g}{v^2}\right)\right\}$$

with $\alpha = 8.10(10^{-3})$, $g = 32.2$ fps^2, $\beta = 0.74$ and Φ is the N(0,1) normal d.f. [3,4] (for justification of the formula for $\sigma_W^2(v)$ see Appendix III). Set

$$\sigma^* = \max_{0 \leq v \leq 45} \sigma_W(v) \text{ , then } \sigma^* = 4.854 \text{ fps} \qquad (8)$$

The cutoff point 45 is used since $P([\underline{U}<45]) = 1$. In (iv) and (v) we will write

$$P(\underline{\theta}<y|\underline{U} = v) = P(\underline{\theta}(v)<y)^{(3)}$$

and

$$P(\underline{W}<y|\underline{U} = v) = P(\underline{W}(v)<y) \text{ , respectively} \quad .$$

---

(3)Then, for example,

$$P(\underline{\theta}(v)<y) = \begin{cases} 2\Phi\left(\dfrac{y}{\sigma_S(v)}\right) - 1 , & y > 0 \\ 0 , & y \leq 0 \end{cases}$$

from (iv).

We assume that

(vi) $$P([\underline{\theta}(v)<y_1,\underline{W}(v)<y_2]) = P([\underline{\theta}(v)<y_1])P([\underline{W}(v)<y_2]) \; ;$$

that is, the r.v.'s $\underline{W}$ and $\underline{\theta}$ are "conditionally independent".

Note that $\underline{W}(v)$ does not mean the value of the r.v. $\underline{W}$ at v; to this end we observe that, e.g., $[W(v)<y_2] = \{e:(\underline{W}(v))(e)<y_2\}$.

4.0 MATHEMATICAL SOLUTION

Recall the opening remarks of **Section** 2.0 and let

$$B_\xi = \left\{(w,\lambda)\varepsilon R_2 \quad : \quad \ddot{x}(w,\lambda)\geq\xi\right\}$$

for some (fixed) $\xi>0$. Then the calculation of equation (7) is equivalent to the calculation of

$$P\left\{[(\underline{W},\underline{\lambda})\,\varepsilon B_\xi\cap C(\delta_1)]\right\} \tag{9}$$

and

$$P\left\{[(\underline{W},\underline{\lambda})\varepsilon\; C(\delta_1)]\right\} > 0 \tag{10}$$

For this purpose, let $\xi>0$ be fixed and consider the following three cases:

I. $B_\xi\cap C(\delta_1) = \phi$, the empty set

II. $B_\xi\cap C(\delta_1) = C(\delta_1)$

III. $\phi \neq B_\xi\cap C(\delta_1) \neq C(\delta_1)$.

It is clear that in cases I and II the probability in (7) has the values 0 and 1, respectively. However, this fact is of value only if we know for which values of $\xi>0$ cases I and II occur. Further, case III cannot be considered until the exact form of $B_\xi \cap C(\delta_1)$ is known for various values of ξ. To accomplish this, we set $\overset{\shortparallel}{x} = \xi(\xi>0)$ in (3) and obtain the following equation for λ in terms of w:

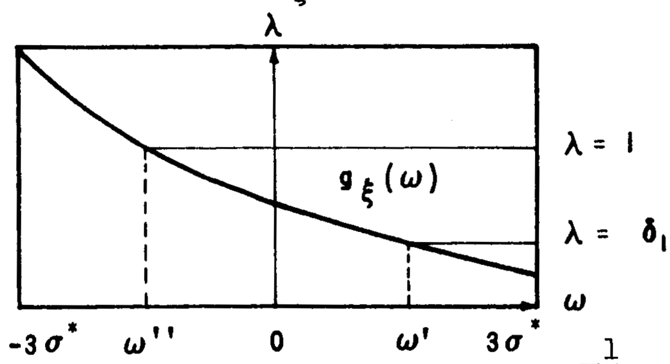
$$g_\xi(w) = \left\{\left(\frac{\xi}{38}\right)\left(\frac{28}{28+w}\right)^{1.3}\right\}^{\frac{1}{14}} , \tag{11}$$

$|w|<3\sigma^*$ where $3\sigma^*<28$. (cf. equation (8)).

Clearly, for fixed $\xi>0$, the mapping $w\rightarrow g_\xi(w)$ $(w>28)$ is a strictly decreasing continuous function such that $g_\xi(w)\rightarrow+\infty$ as $w\rightarrow-28+$ and $g_\xi(w)\rightarrow0$ as $w\rightarrow+\infty$. Hence, the inverse function g_ξ^{-1}, exists and is also strictly decreasing and continuous. Hence, to each $\xi>0$, there corresponds a unique pair of real numbers $(w'w'')$ such that[4]

$$\left.\begin{aligned} w' &= w'(\xi) = g_\xi^{-1}(\delta_1) \\ \text{and} \qquad & \\ w'' &= w''(\xi) = g_\xi^{-1}(1) \quad . \end{aligned}\right\} \tag{12}$$

FOR EXAMPLE:



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(4)Explicitly: $w = g_\xi^{-1}(\delta) = 28\left\{\left(\frac{\xi}{38\delta^{14}}\right)^{\frac{1}{1.3}} - 1\right\}$

Set

$$\left.\begin{aligned} w_1 &= w_1(\xi) = \max\{-3\sigma^* , w''\} \\ w_2 &= w_2(\xi) = \min\{3\sigma^* , w'\} \end{aligned}\right\} \tag{13}$$

and note that, although

$$w'' = g_\xi^{-1}(1) < g_\xi^{-1}(\delta_1) = w' \tag{14}$$

for all $\xi>0$ (since $1 > \delta_1$ and $g_\xi \downarrow$), we may have

$$w_1(\xi) \geq w_2(\xi) \tag{15}$$

for some set of $\xi>0$. By (13), (15) occurs if and only if either

$$w' = w'(\xi) \leq -3\sigma^* \tag{16}$$

or

$$w'' = w''(\xi) \geq +3\sigma^* \tag{17}$$

but not both (cf. (14)). Equation (16) and (17) can be solved to obtain the interval

$$I(\delta_1, \sigma^*) = (a_1(\delta_1),\ a_2(1))^{(5)}$$

such that

$$w_1(\xi) < w_2(\xi)\ ,\ \text{if}\ \xi \in I(\delta_1, \sigma^*)\quad . \tag{18}$$

Further, (16) is equivalent to $\xi \leq a_1(\delta_1)$ and case II. Hence,

$$B_\xi \cap C(\delta_1) = C(\delta_1)\ ,\ \text{if}\ \xi \leq a(\delta_1)\quad . \tag{19}$$

In the same way, (17) is equivalent to $\xi \geq a_2(1)$ and case I. Hence,

$$B_\xi \cap C(\delta_1) = \phi\ ,\ \text{if}\ ,\ \xi \geq a_2(1)\quad . \tag{20}$$

Finally, case III occurs if $\xi \in I(\delta_1, \sigma^*) = (a_1(\delta_1), a_2(\delta_2))$ and in this case

$$B_\xi \cap C(\delta_1) = B_\xi^{(1)} \cup B_\xi^{(2)}\quad , \tag{21}$$

where

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(5) $a_1(\delta_1) = 38 \cdot \delta_1{}^{14}\left(1 - \frac{3\sigma^*}{28}\right)^{1.3}$ $(0 < 3\sigma^* < 28)$ and $a_2(1) = 38\left(1 + \frac{3\sigma^*}{28}\right)^{1.3}$.

$$B_{\xi}^{(1)} = \left\{(w,\lambda);\ w_1(\xi) \le w \le w_2(\xi);\ g_{\xi}(w) \le \lambda \le 1\right\} \tag{22}$$

and

$$B_{\xi}^{(2)} = \left\{(w,\lambda):\ w_2(\xi) \le w \le 3\sigma^*;\ \delta_1 \le \underline{\lambda} < 1\right\}. \tag{23}$$

(If $w_2(\xi) = 3\sigma^*$, set $B_{\xi}^{(2)} = \phi$.)

Hence, since

$$P\left\{[\underline{\ddot{x}}>\xi]\,\middle|\,(\underline{W},\underline{\lambda})\ \varepsilon\ C(\delta_1)\right\} = 0\ ,\ \text{if}\ \xi \ge a_2(1) \tag{24}$$

and

$$P\left\{[\underline{\ddot{x}}>\xi]\,\middle|\,(\underline{W},\underline{\lambda})\ \varepsilon\ C(\delta_1)\right\} = 1\ ,\ \text{if}\ \xi \le a_1(\delta_1)\ , \tag{25}$$

we need only calculate this conditional probability for all $\xi\ \varepsilon\ (a_1(\delta_1),\ a_2(1))$.

Now, $\underline{\lambda}$ is dependent upon the wind speed since $\lambda = \lambda(\gamma)$ and $\gamma = \gamma(\theta(v),\ \alpha,\ \phi)$, so using assumptions (iii - vi) and the Radon-Nikodym definition of conditional probability we obtain

$$P\left\{[(\underline{W},\underline{\lambda})\varepsilon D]\right\} = \int_{-\infty}^{+\infty} P\left\{[(\underline{W}(v),\underline{\lambda}(v))\varepsilon D]\right\}dG(v)$$

$$= \int_{-\infty}^{+\infty}\left\{\iint_D dF_{\underline{\lambda}(v)}^{(\lambda)}\ d\Phi\left|\frac{w}{\sigma_W(v)}\right|\right\}dG(v)\ , \tag{26}$$

where $F_{\underline{\lambda}(v)}$ is the conditional d.f. of the r.v. $\underline{\lambda}$ given that U assumes the value v and, as mentioned in assumption (v), $\Phi\left(\frac{\cdot}{\sigma_W(v)}\right)$ is the conditioned d.f. of the r.v. W given that U = v.

Then, from (21) for

$$\xi \varepsilon (a_1(\delta_1) \, , a_2(1)) \quad , \tag{27}$$

we have

$$\int_{B_\xi^{(1)}} \cup \int_{B_\xi^{(2)}} dF_{\underline{\lambda}(v)}(\lambda) \, d\Phi\left(\frac{w}{\sigma_W(v)}\right) = \int_{w_1(\xi)}^{w_2(\xi)} \left\{ \int_{g_\xi(w)}^{1} dF_{\underline{\lambda}(v)}(\lambda) \right\} d\Phi\left(\frac{w}{\sigma_W(v)}\right)$$

$$+ \int_{w_2(\xi)}^{3\sigma^*} \left\{ \int_{\delta_1}^{1} dF_{\underline{\lambda}(v)}(\lambda) \right\} d\Phi\left(\frac{w}{\sigma_W(v)}\right) \tag{28}$$

It follows easily that

$$P\left\{[\ddot{\underline{X}} \geq \xi][(\underline{W},\underline{\lambda})\varepsilon C(\delta_1)]\right\} = \int_{-\infty}^{+\infty} \left\{ \Phi\left(\frac{3\sigma^*}{\sigma_W(v)}\right) - \Phi\left(\frac{w_1(\xi)}{\sigma_W(v)}\right) - \right.$$

$$- \int_{w_1(\xi)}^{w_2(\xi)} \left(\Phi\left(\frac{w}{\sigma_W(v)}\right)\right)' F_{\underline{\lambda}(v)}(g_\xi(w)) \, dw -$$

$$\left. - \left[\Phi\left(\frac{3\sigma^*}{\sigma_W(v)}\right) - \Phi\left(\frac{w_2(\xi)}{\sigma_W(v)}\right)\right] F_{\underline{\lambda}(v)}(\delta_1) \right\} dG(v) \quad , \tag{29}$$

where we have used the fact that $F_{\underline{\lambda}(v)}(1) = 1$ $(v \geq 0)$ and (27) holds.

In the same manner, it is found that

$$P[(\underline{W},\underline{\lambda})\varepsilon C(\delta_1)] = \int_{-\infty}^{+\infty}\left\{\left(\Phi\left(\frac{3\sigma^*}{\sigma_W(v)}\right) - \Phi\left(\frac{-3\sigma^*}{\sigma_W(v)}\right)\right)(1-F_{\underline{\lambda}(v)}(\delta_1))dG(v)\right. , \tag{30}$$

independent of ξ.

Equations (29) and (30) then allow the computation of $P\{[\underline{\ddot{X}}>\xi]|(\underline{W},\underline{\lambda})\varepsilon C(\delta_1)\}$ when (27) holds, once the functional form of $F_{\underline{\lambda}(v)}$ is known. For this purpose we recall equation (5) in the form

$$\underline{\lambda}(v) = \sin\underline{\theta}(v)\ \sin(\theta_0+\underline{\alpha})\ \sin\underline{\phi} + \cos\underline{\theta}(v)\ \cos(\theta_0+\underline{\alpha}) \underset{\text{Def}}{=} g(\underline{\phi},\underline{\theta}(v),\underline{\alpha}) . \tag{31}$$

Then, since $\underline{\theta}(v)$, $\underline{\phi}$ and $\underline{\alpha}$ are independent r.v.'s,

$$P\left\{[g\ (\underline{\phi},\underline{\theta}(v),\underline{\alpha})<z]|(\underline{\theta}(v),\underline{\alpha}) = (\theta',\alpha')\right\}$$

$$= P\left\{[g(\underline{\phi},\theta',\alpha')<z]|(\underline{\theta}(v),\underline{\alpha}) = (\theta',\alpha')\right\} = P\left\{[g(\underline{\phi},\theta',\alpha')<z]\right\}. \tag{32}$$

Hence, from the definition of conditional probability and the independence of $\underline{\theta}(v)$ and $\underline{\alpha}$,

$$F_{\underline{\lambda}(v)}(z) = P(g(\underline{\phi},\underline{\theta}(v),\underline{\alpha})<z)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P([g(\underline{\phi},\theta',\alpha')<z])dF_{\underline{\theta}(v)}(\theta')dF_{\underline{\alpha}}(\alpha') , \tag{33}$$

where $F_{\underline{\theta}(v)}$ and $F_{\underline{\alpha}}$ are the d.f.'s of $\underline{\theta}(v)$ and $\underline{\alpha}$, respectively.

Finally, using assumptions (i) and (iii) it is easily seen (and well-known) that

$$F_{\underline{\alpha}}(y) = \begin{cases} 0 & , \quad y \leqq -\alpha_o \\ \frac{1}{\pi}\left\{\sin^{-1}\left(\frac{y}{\alpha_o}\right) + \frac{\pi}{2}\right\} & , \quad |y| < \alpha_o \\ 1 & , \quad y \geqq \alpha_o \end{cases} \tag{34}$$

and

$$F_{\underline{\theta}(v)}(y) = \begin{cases} 2\Phi\left(\frac{y}{\sigma_s(v)}\right) - 1 & , \quad y > 0 \\ 0 & , \quad y \leqq 0 \quad , \end{cases} \tag{35}$$

where Φ is the N(0,1), normal d.f.

Let

$$\left.\begin{aligned} A(\theta,\alpha) &= \sin\theta \sin(\theta_o+\alpha) \\ B(\theta,\alpha) &= \cos\theta \cos(\theta_o+\alpha) \end{aligned}\right\} \quad . \tag{36}$$

Then (33) becomes

$$F_{\underline{\lambda}(v)}(z) = \int_{-\alpha_0}^{\alpha_0} \frac{2}{\pi} \int_0^{\infty} P(A(\theta,\alpha)\ \sin\underline{\phi} < z - B(\theta,\alpha)) \quad \frac{d\Phi\left(\frac{\theta}{\sigma_s(v)}\right) d\alpha}{\sqrt{\alpha_o^2 - \alpha^2}} , \tag{37}$$

where the d.f. of the r.v. $\sin\underline{\phi}$ is well-known.

Combining equations (24), (25), (29), (30) and (37) we have the complete solution of the problem under the stated assumptions.

5.0 THE NUMERICAL CALCULATION OF $P(\underline{\ddot{X}} \geq \xi | (\underline{W},\underline{\lambda})\varepsilon C(\delta_1))$

Due to the fact that the d.f. G of the wind speed is known only in graph form (Appendix IV), we replace G by the step function

$$F_u(v) = \sum_{\ell=1}^{M} g_\ell\ \varepsilon(v-v_\ell)^{(6)} , \tag{38}$$

where the v_ℓ ($\ell=1,2,\cdots,M$) are M suitably chosen velocities between 0 and 45 knots and

$$g_\ell = G(v_\ell) - G(v_{\ell-1}) ,(\ell=1,2,\cdots,M) ,$$

with $v_o=0$, $v_M=45$ knots. Since the original graph of G was constructed by smoothing a finite number of experimental observations,

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(6) ε is the unit d.f. taking one jump of magnitude 1 at the origin.

this is not much of a restriction. Even so, by taking M large we can make the agreement between the hypothetical G and F_u as close as desired. However, as we shall point out below the magnitude of M will be restricted by computer time.

As for the simplifications to be introduced below they can be thought of as leading to approximations of the theoretical expressions in equations (29) and (37); or they can be considered as simplifications in the original model. The latter viewpoint is by far the most honest; indeed, the former interpretation tacitly assumes that we are able to make the approximation differ from the true, but unknown, value by at most a specified error, and this it seems requires far too much machine time to be feasible.

The first such simplification, consists in "grouping the data" concerning the wave slope. That is, we replace the absolutely continuous d.f. of $\underline{\theta}(v)$ by the step function

$$F_{\tilde{\underline{\theta}}(v)}(y) = \sum_{n=1}^{N} p_n^{(v)} \; \varepsilon(y-\theta_n) \quad , \tag{39}$$

where $\theta_n = n\delta_N$ for $n=1,2,\cdots,N-1$ with $\delta_N = (19/N-1)°$, $\theta_N = 18.5°$[7], and where

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(7)The choice of the cutoff point 18.5° is reasonable since $\sigma_s(45) = \max\limits_{0<v<45} \sigma_s(v) = .1381032$ radians and $2\Phi\left(\frac{(18.5)°}{\sigma_s(v)}\right) - 1 \cong 0.98$.

$$p_n^{(v)} = P((n-1)\delta_N \leqq \underline{\theta}(v) < n\delta_N) = 2\left[\Phi\left(\frac{n\delta_N}{\sigma_s(v)}\right) - \Phi\left(\frac{(n-1)\delta_N}{\sigma_s(v)}\right)\right]$$

for $n = 1,2,\cdots,N-1$ and

$$p_N^{(v)} = P(\underline{\theta}(v) \geqq 18°) = 2\left[1-\Phi\left(\frac{18°}{\sigma_s(v)}\right)\right] \quad .$$

Introducing equations (38) and (39) into (29), (30) and (37), setting

$$\left.\begin{aligned} A_n(\alpha) &= \sin\,\theta_n\,\sin(\theta_o+\alpha) \\ \\ B_n(\alpha) &= \cos\,\theta_n\,\cos(\theta_o+\alpha) \end{aligned}\right\} \tag{40}$$

and noting that

$$A_n(\alpha) > 0 \quad \text{for all} \quad \alpha\varepsilon(-\alpha_o,\alpha_o)$$

we obtain

$$P([\underline{\ddot{X}} \geq \xi] \,|\, (\underline{W},\underline{\lambda})\,\varepsilon C(\delta_1)) = \frac{\sum_{\ell=1}^{M} P(A_\xi^{(\ell)}(\delta_1))g_\ell}{Q(\delta_1)} \tag{41}$$

$\xi\varepsilon(a_1(\delta_1),a_2(1))$, where

$$P(A_\xi^{(\ell)}(\delta_1)) = \left[\Phi\left(\frac{3\sigma^*}{\sigma_w(v\ )}\right) - \Phi\left(\frac{w_1(\xi)}{\sigma_w(v_\ell)}\right)\right] -$$

$$- \frac{1}{\pi} \sum_{n=1}^{N} p_n^{(v_\ell)} \int_{w_1(\xi)}^{w_2(\xi)} \left(\Phi\left(\frac{w}{\sigma_w(v_\ell)}\right)\right)' \int_{-\alpha_o}^{\alpha_o} F_{\sin\phi}\left(\frac{g_\xi(v)-B_n(\alpha)}{A_n(\alpha)}\right) \frac{d\alpha\ dw}{\sqrt{\alpha_o^2 - \alpha^2}} -$$

$$- \left[\Phi\left(\frac{3\sigma^*}{\sigma_w(v_\ell)}\right) - \Phi\left(\frac{w_2(\xi)}{\sigma_w(v_\ell)}\right)\right] \cdot \frac{1}{\pi} \sum_{n=1}^{N} p_n^{(v_\ell)} \int_{-\alpha_o}^{\alpha_o} F_{\sin\phi}\left(\frac{\delta_1 - B_n(\alpha)}{A_n(\alpha)}\right) \frac{d\alpha}{\sqrt{\alpha_o^2 - \alpha^2}} ,$$

$$\ell = 1,2,\cdots,M \quad , \qquad (42)$$

where $F_{\sin\phi}$ is the d.f. of $\sin\phi$ (an "arc-sine law").

$Q(\delta_1)$ is given by a slightly simpler expression of the same type derived from equation (30).

Two problems arise with the calculation of (41) via (42): machine time and accuracy. If the accuracy of the calculation of the double integral is brought within the desired limit, then the machine time is prohibitive even for moderate values of M and N; e.g., if $M = 10$, $N = 20$ and the double integral requires 2 minutes, then just this part of the program requires over six hours and this is just for one value of ξ! Moreover, at $\xi = 20°$, the normal mission limit, the third term in (42) is zero and the sums with respect to ℓ of the first two terms are each very close to one; hence, errors in the numerical value of the sum (relative to n) of the N double integrals in even the second or third decimal place are serious.

In order to facilitate the calculation and to obtain results which will be more accurately interpreted, the model is once more simplified to the extent that the d.f. $F_{\underline{\alpha}}$ of the sinusoidal oscillation is "quantized". This introduces only single integrals into the final calculation and the machine is capable of handling these with rapidity and accuracy.

Hence, in place of $F_{\underline{\alpha}}$ in (33), we use

$$F_{\underline{\tilde{\alpha}}}(y) = \sum_{i=1}^{R} h_i \; \varepsilon(y-x_i) \quad ,$$

where

$$x_i = -\alpha_o + (i-1)\delta_R \quad \text{for } i = 2,3,\cdots,R$$

with

$$\delta_R = 2\alpha_o/(R-1) \;, \; x_1 = -(3.8)^\circ$$

and

$$h_i = P(x_{i-1} \leqq \alpha < x_i) = F_{\underline{\alpha}}(x_i) - F_{\underline{\alpha}}(x_{i-1})$$

for $i = 2,3,\cdots,R$ and

$$h_1 = F_{\underline{\alpha}}(-(3.8)^\circ) \quad .$$

Therefore, the final results for $a_1(\delta_1) < \xi < a_2(1)$ are based on equation (41) with (42) replaced by

$$P(A_\xi^{(\ell)}(\delta_1)) = \left[\Phi\left(\frac{3\sigma^*}{\sigma_w(v_\ell)}\right) - \Phi\left(\frac{w_1(\xi)}{\sigma_w(v_\ell)}\right)\right] -$$

$$- \sum_{n=1}^{N} p_n^{(v_\ell)} \sum_{i=1}^{R} h_i \int_{w_1(\xi)}^{w_2(\xi)} \left(\Phi\left(\frac{w}{\sigma_w(v_\ell)}\right)\right)' F_{\sin\phi}\left(\frac{g_\xi(w)-B_n(x_i)}{A_n(x_i)}\right) dw -$$

$$- \left[\Phi\left(\frac{3\sigma^*}{\sigma_w(v_\ell)}\right) - \Phi\left(\frac{w_2(\xi)}{\sigma_w(v_\ell)}\right)\right] \sum_{n=1}^{N} p_n^{(v_\ell)} \sum_{i=1}^{R} h_i\, F_{\sin\phi}\left(\frac{\delta_1 - B_n(x_i)}{A_n(x_i)}\right)$$

and a similar expression for $Q(\delta_1)$.

6.0 CONCLUSIONS

It has been pointed out earlier (Section 2.0) that the critical values of the acceleration $\ddot{X}$ occur when the entry angle, γ, is "small". Thus, the limitation of having no data above an entry angle of 27° is of little consequence.

Heuristically, this is simply the statement that, for moderate to large values of ξ, the probability that $\ddot{X}$ exceeds ξ g's given that $\gamma > 27°$ is not greater than the probability of this event given that $0 \leq \gamma \leq 27°$. This, in turn, is essentially the statement that

$$P(\underline{\ddot{X}} \geq \xi \,|\, (W,\lambda) \notin C(\delta_1)) \leq P(\underline{\ddot{X}} \geq \xi \,|\, (W,\lambda) \in C(\delta_1)). \quad (43)$$

Moreover, it is easily seen that if (43) holds, then

$$P(\ddot{X} \geq \xi) \leq P(\ddot{X} \geq \xi \,|\, (W,\lambda) \varepsilon C(\delta_1)). \tag{44}$$

It is the engineering consensus that (43) holds at least for $\xi \geq 10$ g's. Thus, the calculated values of $P(\ddot{X} \geq \xi \,|\, (W,\lambda) \varepsilon C(\delta_1))$ constitute an upper bound on the unconditional probability that $\ddot{X} \geq \xi$, for $\xi \geq 10$ g's.

At the time this paper was written the normal mission limit was approximately 20 g's and the emergency mission limit was approximately 40 g's. From the table in Appendix II it is seen that, according to our model and the preceding remarks, the former event should not occur more than 5 times in a series of 100 landings, while the latter event should not occur at all. More detailed results are available in Appendix II.

In addition to using equation (44) in the interpretation of these results, it must be noted that the model constructed here does not take into consideration the crosswind component of the slope of the sea surface (see [1]), but only the up and downwind components. However, the introduction of the crosswind components into the model would only reduce the likelihood of occurrence of waves with "large" slopes and, hence, decrease the probability of the event $\ddot{X} \geq \xi$ as compared to this probability in the given model. Thus, with respect to sea surface slopes, our probabilities of the value of the acceleration are somewhat larger than should be expected in reality.

In this direction, it should also be pointed out that in the approximation of the graph of $\ddot{X}(0,\gamma)$ (see Section 1.0, p. 4) by 38 $(\cos\gamma)^{14} = g(\gamma)$, we have a uniform overestimate, i.e., $\ddot{X}(0,\gamma) \leqq g(\gamma)$ for all γ, $0 \leqq \gamma \leqq 27°$. This fact has the consequence that once again the probabilities generated by our model are slightly in excess of those that should be expected in reality.

Finally, the analysis for which the above information was requested also required the probability that the entry angle, γ, will exceed a given angle θ. This follows immediately from equation (33) and the remarks in Section 5.0. Calculation yields the following table:

θ	$P(\gamma \geqq \theta)$
5°	1.000
10°	0.998
15°	0.985
20°	0.932
25°	0.684
30°	0.326
31°	0.237
35°	0.094
40°	0.011
45°	0.001

It is instructive to notice that on the basis of this table the events $[0 \leqq \gamma \leqq 27°]$ and $[27° \leqq \gamma]$ are approximately equally probable and, further, to point out that this has nothing to do with the implication $(43) \Longrightarrow (44)$.

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1033-GRA-jdc G. R. Andersen

Attachments
References
Appendices I - IV

REFERENCES

[1] CM Water Landing Criteria - Ref. P6-5/L114/65 - 746; letter sent by MSC to North American Aviation Company on July 6, 1965.

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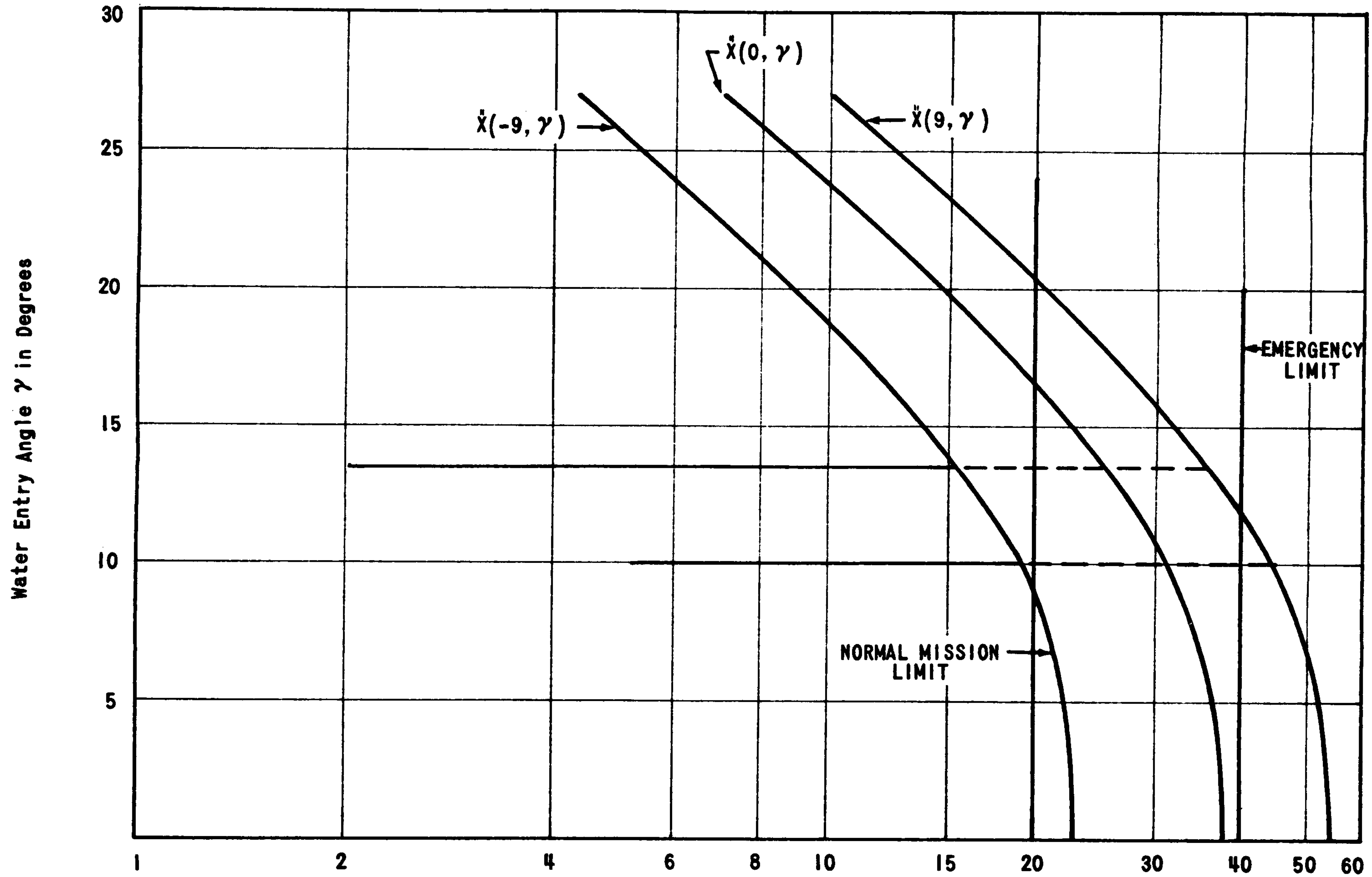
[3] W. J. Pierson, Jr., Lionel Maskowitz, "A Proposed Spectral Form for Fully Developed Wind Seas based on the Similarity Theory of S. A. Kitaigorodskii," Journal of Geophysics Research, 96, 24 5181-5190.

[4] Letter from D. E. Cartwright (National Institute for Oceanography, Wormley (Surrey) 2122, England) to W. W. Elam, Bellcomm, Inc.

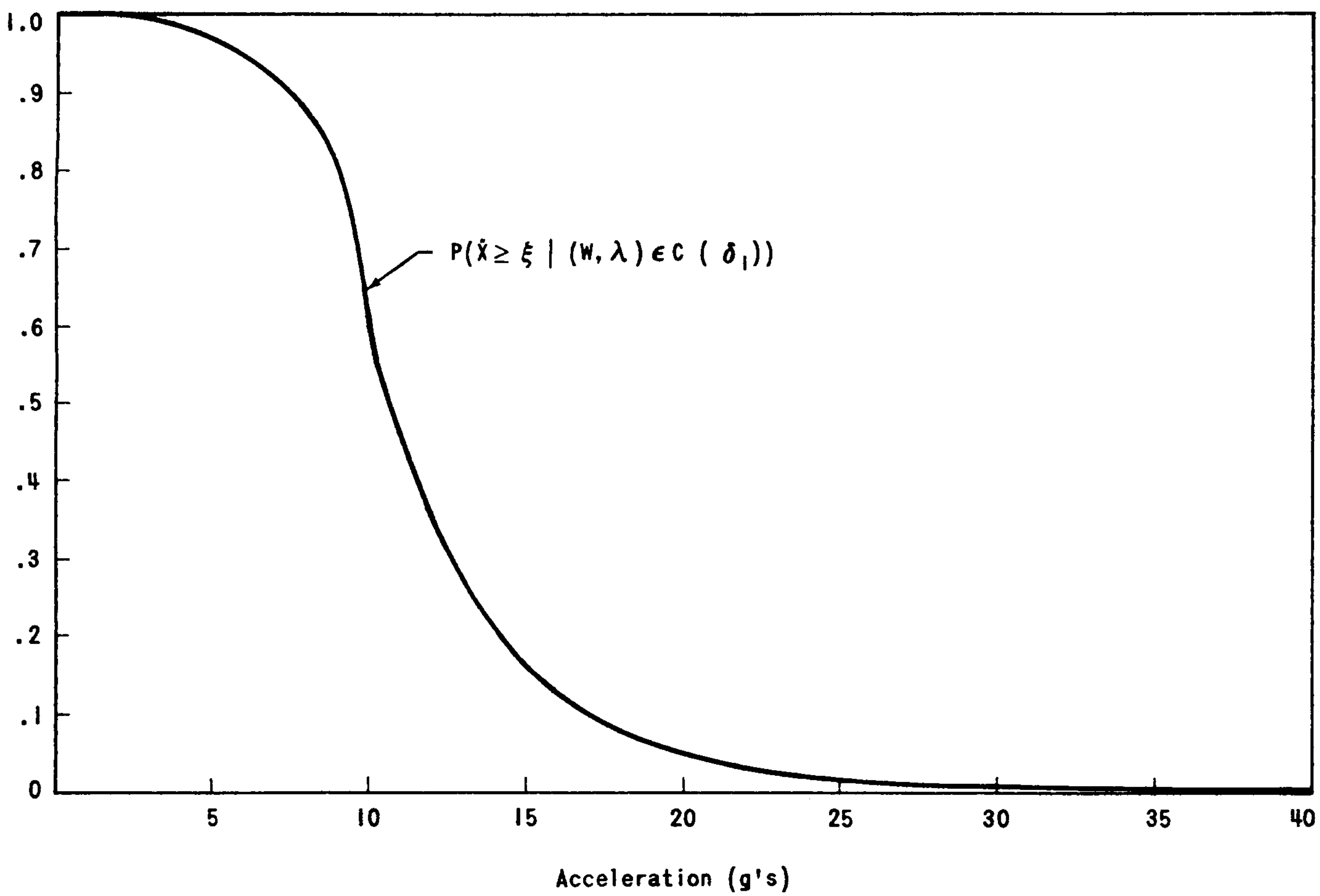
APPENDIX I



$\ddot{X} = \ddot{X}(W, \gamma)$ in g's, Where W Denotes Water Particle Velocity in fps

CALCULATED CM WATER LANDING IMPACT ACCELERATIONS FOR 3 PARACHUTE CLUSTER

APPENDIX II



ξ	$P(\ddot{X} \geq \xi \mid (W, \lambda) \in C(\delta_1))$
2	1
8	0.877
10	0.591
10.5	0.524
11	0.462
12	0.357
15	0.163
18	0.079
20	0.049
22	0.031
25	0.016
30	0.005
35	0.001
40	0.000

APPENDIX III

In references [2] and [4], we obtained

$$\sigma_w^2(v) = \alpha g^2 \int_{w_1}^{w_2} \frac{1}{w^3} e^{-\beta \left(\frac{w_o}{w}\right)^2} dw, \qquad (*)$$

where $w_o = g/v$, v is the wind speed in fps, $g = 32.2$ fps^2, $\alpha = 8.10\ (10^{-3})$, $\beta = 0.74$ (dimensionless constants) and $w_i = \sqrt{gk_i}$ $(i = 1,2)$ with $k_i = 2\pi/\lambda_i$, where λ_i is the wave length in feet.

Under a simple transformation equation (*) becomes

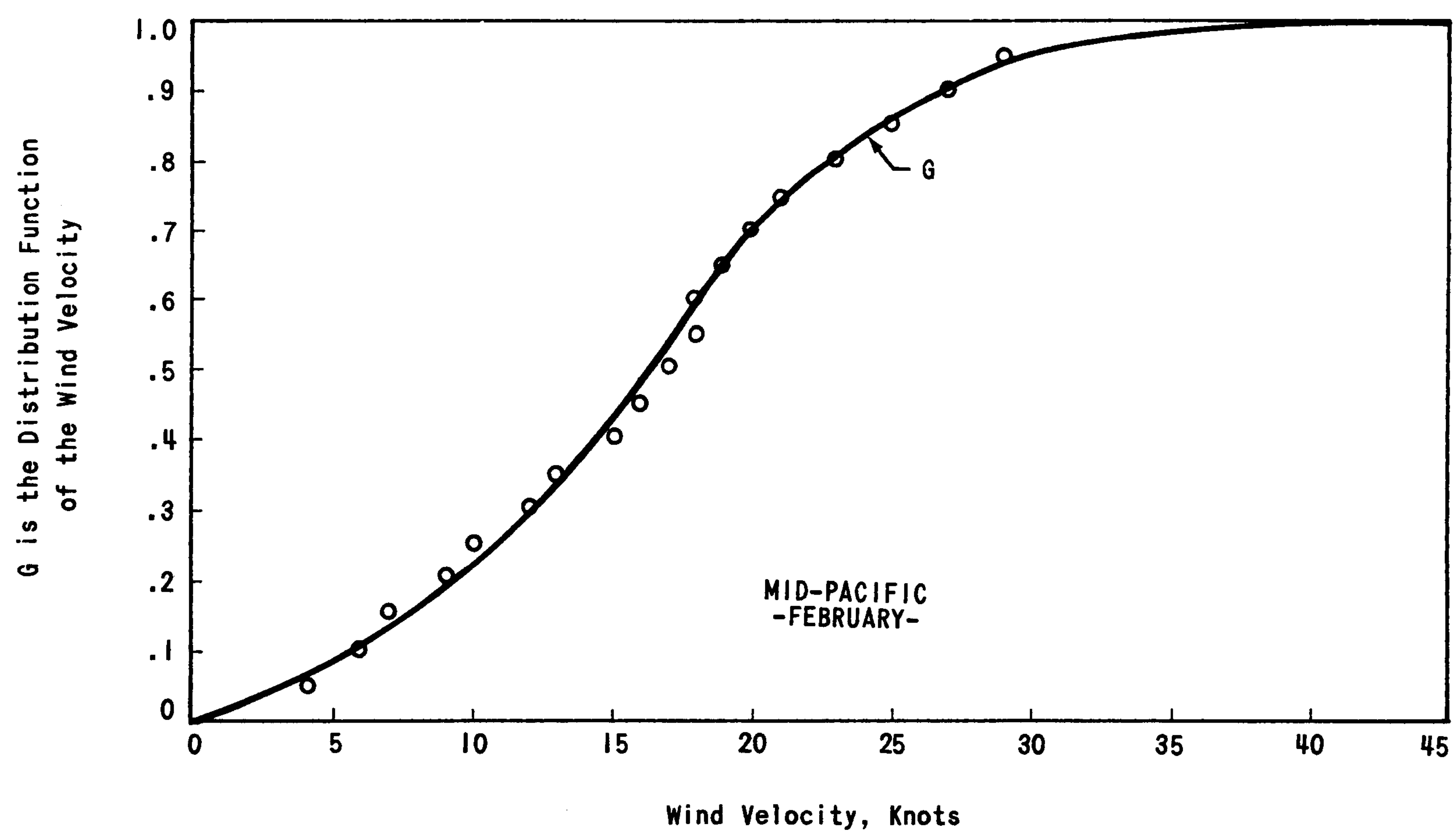
$$\sigma^2(v) = \frac{\alpha v^2}{2} \sqrt{\frac{\pi}{\beta}} \left\{ \Phi\left(\sqrt{2\beta}\left(\frac{g}{w_1 v}\right)^2\right) - \Phi\left(\sqrt{2\beta}\left(\frac{g}{w_2 v}\right)^2\right)\right\}.$$

Now, the agency requesting this study required that $\lambda_1 = +\alpha$ and $\lambda_2 = 26$. Hence, we obtain

$$\sigma^2(v) = \frac{\alpha v^2}{2} \sqrt{\frac{\pi}{\beta}} \left\{ 1 - \Phi\left(\frac{13}{\pi} \sqrt{2\beta}\ \frac{g}{v^2}\right) \right\}.$$

The reason for expressing $\sigma^2(v)$ in terms of the normal d.f. is of course to avoid the tedious numerical integration of the integral in equation (*); since the normal d.f. is well tabulated.

APPENDIX IV


G is the Distribution Function
of the Wind Velocity
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
5
10
15
20
25
30
35
40
45
Wind Velocity, Knots
G
MID-PACIFIC
-FEBRUARY-